BIBLIOGRAPHY

1. Barmin, A. A. and Kulikovskii, A. G., Gas velocity variation in ionizing shock waves. The problem of the conductive piston. PMM Vol. 32, №3, 1968.
2. Miller, B., Experimental study of normal ionizing shock waves. Phys. Fluids, Vol. 10, №1, 1967.
3. Heywood, J., Experiments in a magnetically driven shock tube with an axial magnetic field. Phys. Fluids, Vol. 9, №6, 1966.
4. Taussing, R. T., Normal ionizing shock waves. Phys. Fluids, Vol. 8, №9, 1965.
5. Barmin, A. A. and Kulikovskii, A. G., On shock waves ionizing a gas in a magnetic field. Dokl. Akad. Nauk SSSR, Vol. 178, №1, 1968.
6. Kulikovskii, A. G. and Liubimov, G. A., Magnetohydrodynamics, M., Fizmatgiz, 1962.

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ON THE EVOLUTIONARITY OF EQUATIONS OF MAGNETOHYDRODYNAMICS TAKING THE HALL EFFECT INTO ACCOUNT

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It has been established that the system of equations of magnetohydrodynamics for a nondissipative plasma plane flow across a magnetic field will not be evolutionary if the Hall effect is taken into account.

The numerical solution of the problem of plasma flow in a coaxial channel with the Hall effect taken into account had disclosed [1] the flow instability which is the more pronounced the stronger the Hall effect is. This instability of a nonstationary plasma flow develops in the vicinity of the anode, and has the character of an explosion in which a sharp rise of the current density and particle velocity takes place. An experimental investigation of such flows [2] had disclosed the appearance of considerable jumps of potential in the anode vicinity with the flow itself becoming unstable, resulting in the so-called "current attachment" and severe anode erosion.

These results and observations make it desirable to carry out a mathematical analysis of the two-dimensional plasma flow stability taking into account the Hall effect. The present paper is devoted to a comparatively simpler, but important result which has to be kept in mind in detailed investigations on this subject, namely that the equations of nondissipative magnetohydrodynamics for a two-dimensional plane flow across a magnetic field are nonevolutionary when the Hall effect is taken into account. The term nonevolutionary (or incorrectly) [3] is understood here to mean the instability of solution of the Cauchy problem with respect to high frequency perturbations which increase arbitrarily fast. This result is also true for axisymmetric flows acted upon by an azimuthal magnetic field, in so far as these may be considered to be locally plane.

The flow of an inviscid and non-heat-conductive plasma in the presence of the Hall effect is defined (in a dimensionless form and with the usual notations [1]) by Eqs.

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0 \qquad \xi = \frac{c}{eL}\left(\frac{M}{4\pi\rho_0}\right)^{1/2}$$
$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + [\operatorname{rot} \mathbf{H} \times \mathbf{H}] \qquad \left(\eta = \frac{c^2}{4\pi\sigma L v_0} = \frac{1}{R_m}\right) \tag{1}$$
$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}(\mathbf{v}\times\mathbf{H}) - \xi \operatorname{rot}\left[\left(\frac{\operatorname{rot}\mathbf{H}}{\rho}\times\mathbf{H}\right)\right] - \operatorname{rot}(\eta \operatorname{rot}\mathbf{H})$$

This system must be complemented by the adiabatic condition (or the isothermal condition when $\gamma = 1$)
$$p = \beta\rho^{\gamma} \qquad \left(\beta = \frac{4\pi p_0}{H_0^2}\right)$$

Here β is the characteristic for this problem ratio of the gas and magnetic pressures, ξ is the exchange parameter characterizing the Hall effect, and η a dimensionless parameter which is the reciprocal of conductivity.

In the case of a plane flow across a magnetic field $v_z = H_x = H_y = 0$, and in addition $\partial(\ldots)/\partial z = 0$. Eqs. (1) are of the form
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$
$$\rho\frac{du}{dt} = -\frac{\partial}{\partial x}\left(p + \frac{H^2}{2}\right), \qquad \rho\frac{dv}{dt} = -\frac{\partial}{\partial y}\left(p + \frac{H^2}{2}\right) \tag{2}$$
$$\frac{\partial H}{\partial t} + \frac{\partial Hu}{\partial x} + \frac{\partial Hv}{\partial y} + \xi\frac{H}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial\rho}{\partial y}\frac{\partial H}{\partial x}\right) = \eta\Delta H$$

Here
$$u = v_x, \quad v = v_y, \quad H = H_z, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

A singularity of this type of flow is the degenerate character of the Hall effect. It appears in Eq. (2) in the form of the product of first derivatives, while in accordance with (1) the principal terms of the corresponding degeneration must, generally speaking, be of the second order.

In the analysis of stability with respect to high frequency perturbations system (2) is usually linearized, and its coefficients assumed constant. In order to assess the influence of the Hall effect it is necessary, when linearizing the Hall term in (2), to take into account the derivatives of the unperturbed solution, and to assume these to be constant. With perturbations denoted by ρ_1, u_1, v_1 and H_1 the linearized system of equations is as follows:

$$\frac{d\rho_1}{dt} + \rho\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = 0 \qquad \frac{du_1}{dt} + \frac{C^2}{\rho}\frac{\partial\rho_1}{\partial x} + \frac{H}{\rho}\frac{\partial H_1}{\partial x} = 0 \qquad \left(C^2 = \frac{dp}{d\rho} = \beta\gamma\rho^{\gamma-1}\right)$$
$$\frac{dv_1}{dt} + \frac{C^2}{\rho}\frac{\partial\rho_1}{\partial y} + \frac{H}{\rho}\frac{\partial H_1}{\partial y} = 0, \qquad \frac{dH_1}{dt} + H\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) +$$
$$+ \xi\frac{H}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial H_1}{\partial y} - \frac{\partial\rho}{\partial y}\frac{\partial H_1}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial\rho_1}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial\rho_1}{\partial y}\right) = \eta\Delta H_1 \tag{3}$$

Here all coefficients including $\partial\rho/\partial x$, $\partial\rho/\partial y$, $\partial H/\partial x$, $\partial H/\partial y$ are constant.

Solutions of system (3) are sought in the form of plane waves, i.e. $\exp(i\omega t + ik_1x + ik_2y)$ with constant multipliers, where k_1, k_2 are arbitrary real numbers (see, e.g. [3]). We introduce notations $s^2 = k_1^2 + k_2^2$, $\omega = s\lambda$, $k_1 = s\mu$, $k_2 = s\nu$. Then the direction of the plane wave will be defined by the unit vector $\mathbf{l} = (\mu, \nu)$, and it will be natural to write
$$\mu u + \nu v = v_l, \quad \mu\partial/\partial y - \nu\partial/\partial x = \partial/\partial\tau$$
where τ is the direction along the wave front orthogonal to $\mathbf{l}$. Finally, denoting

$\lambda + v_l = z$, we obtain the characteristic equation binding ω, k_1 and k_2 in the simple form

$$z\left(z^2 - \frac{\xi H}{\rho^2}\frac{\partial \rho}{\partial \tau} z^2 - C_m^2 z + \frac{\xi H}{\rho^2}\frac{\partial P}{\partial \tau} - is\eta\,(z^2 - C^2)\right] = 0 \qquad (4)$$

$$(C_m^2 = C^2 + H^2/\rho, \ P = p + H^2/2)$$

The condition of correctness of the Cauchy problem for systems (2) and (3) for any real μ, ν and $s > 0$ is obviously the inequality

$$\operatorname{Im} \omega = s \operatorname{Im} \lambda = s \operatorname{Im} z \geqslant \text{const} \qquad (5)$$

For the finite values of s the solutions are always finite, hence in practice the fulfilment of condition (5) for $s \to \infty$ must be required.

It is easily ascertained that when $\eta \neq 0$, then the asymptotic behavior of roots of Eq. (4) for $s \to \infty$ is defined by $z_1 = is\eta + O(1)$, $z_{2,3} = \pm C + O(1/s)$

Magnitudes $s > 0$ and $\eta > 0$, hence they satisfy condition (5), i.e. for finite conductivity the problem here considered is evolutionary (correct).

In the case of perfect conductivity, i.e. when $\eta = 0$, Eq. (4) has real coefficients independent of s. If it has two complex conjugate roots, then for one of these $\operatorname{Im} z < 0$, and $\operatorname{Im} \omega \to -\infty$ when $s \to \infty$, i.e. condition (5) is not fulfilled. This means that the fulfillment of condition (5), i.e. the correctness of the problem, is equivalent to the requirement for all three roots of Eq. (4) to be real.

As is known, all roots of Eq. $z^3 - pz^2 + qz - r = 0$ with real coefficients are real, if and only if the discriminant

$$D \equiv p^2q^2 + 18pqr - (4p^3r + 4q^3 + 27r^2) \geqslant 0$$

In the case of Eq. (4) (for $\eta = 0$) this becomes

$$D = 4gX^2 + (C_m^4 + 18C_m^2 g - 27g^2)\,X + 4C_m^6 \qquad (6)$$

$$X = \left(\frac{\xi H}{\rho^2}\frac{\partial \rho}{\partial \tau}\right)^2, \qquad g = \frac{\partial P/\partial \tau}{\partial \rho/\partial \tau}$$

An elementary analysis of the quadratic trinomial (6) will show that $D \geqslant 0$ for all $X > 0$, if and only if

$$0 < g \leqslant C_m^2 \qquad (7)$$

This means that when condition (7) is fulfilled, then the system of Eqs. (2) is evolutionary for all possible directions of vector τ. If it is violated, then Eq. (4) has complex roots which lead to incorrectness in the following cases:

$$1)\ g < 0, \quad X > X_2; \qquad 2)\ g > C_m^2, \quad X_1 < X < X_2$$

Here X_1 and X_2 are the roots of trinomial D.

In actual plasma flows condition (7) is, as a rule, violated. In fact, if gradients of ρ and $P = p + H^2/2$ are not collinear then a direction in which $g < 0$ will always be found.

Thus in the case of infinite plasma conductivity Eqs. (2) are nonevolutionary. This property of equations is apparently responsible for the appearance of instability in the computation of flows with the Hall effect the sooner the higher the plasma conductivity.

We note in conclusion that the obtained result is due to the degenerate character of the Hall effect in Eqs. (2) of a plane flow across a magnetic field. It is not possible to investigate the correctness of an arbitrary flow by the method of plane waves. A repetition of the reasoning outlined above in the general three-dimensional case (or even for the case of three-dimensional perturbations of the flow here considered) leads to conclusions as follows:

1. For finite conductivity ($\eta \neq 0$) the equations are, as before evolutionary.

2. For perfect conductivity ($\eta = 0$) the characteristic equation generalizing (4) has part of its roots of the form $z = as + O(1)$, i.e. $\omega = as^2 + O(s)$ in which a is real. The predominant term of this expression is "neutral", and the fulfillment of condition (5) for $s \to \infty$ depends on the next following term, while in the method of plane waves, which presupposes constant coefficients, the predominant asymptotic terms only have a meaning.

Thus, in the general nondissipative case the problem is much more involved, necessitating the consideration of variable coefficients of equations. In a way it will be a generalization of the system of equations of the complicated problem of the evolutionary property of the Schroedinger equation with variable coefficients.

BIBLIOGRAPHY

1. Brushlinskii, K. V., Gerlakh, N. I. and Morozov, A. I., Computation of nonstationary two-dimensional flows of plasma of finite conductivity in the presence of the Hall effect. Magnetohydrodynamics, №1, 1967.
2. Kovrov, P. E., Morozov, A. I., Tokarev, L. G. and Shchepkin, G. Ia., Magnetic field distribution in a coaxial plasma injector. Dokl. Akad. Nauk SSSR, Vol. 172, №6, 1967.
3. Gel'fand, I. M., Certain problems of the theory of quasi-linear equations. Usp. Matem. Nauk, Vol. 14, №2, 1959.

Translated by J. J. D.

SIMPLE WAVE IN A CONDUCTIVE MEDIUM IN A STATIONARY GRAVITATIONAL FIELD

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1. We shall consider the simultaneous nonstationary motion of a perfectly conductive gas in a homogeneous magnetic field H perpendicular to the direction of velocity taking into account the force of gravity g. If at the initial instant the motion is isentropic, then with the condition of frozen lines of magnetic field $H = b\rho$ the coefficient b will remain constant throughout the duration of motion.

Then, in the case here considered the Euler equation together with the continuity equation is reduced to the system of conventional gas dynamics equations [1 and 2]

$$u_t + uu_x + \frac{p_m}{\rho} = -g, \qquad \rho_t + u\rho_x + \rho u_x = 0, \qquad p_m = A\rho^k + \frac{H^2}{8\pi} \qquad (1.1)$$

The effective velocity of sound in this case is

$$c_m^2 = \left(\frac{\partial p_m}{\partial \rho}\right)_s = c^2 + v_a^2, \qquad c = (Ak\rho^{k-1})^{1/2}, \qquad v_a = \left(\frac{b^2\rho}{4\pi}\right)^{1/2} \qquad (1.2)$$

Here c is the usual velocity of sound in a conductive medium, and v_a the Alfven